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1995 J. Phys. A: Math. Gen. 28 4381

(http://iopscience.iop.org/0305-4470/28/15/017)

J. Phys. A: Math. Gen. 28 (1995) 4381–4388. Printed in the UK

Relationship between the Moyal KP and the Sato KP hierarchies

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Received 8 December 1994, in final form 3 April 1995

Abstract. The formula describing the isomorphism between the Moyal KP and the Sato KP hierarchies is given. We obtain the conserved densities of the dispersionless KP hierarchy by taking the limit $k \to 0$ of the conserved quantities derived in the Sato approach.

1. Introduction

In the last few years there have been many remarkable developments in the theory of integrable nonlinear differential equations [1]. One of the most investigated systems is the Kadomtsev–Petwiashvili (KP) hierarchy [2] and its various generalizations [3–6]. There are different approaches to the description of the algebraical and geometrical properties of the KP hierarchy, e.g. the Sato theory [7], Hirota's method [8], the r-matrix approach [9] etc.

The Sato theory is based on the treatment of partial differential KP equations as dynamical systems on the infinite-dimensional algebra of pseudo-differential operators (see e.g. [10]). By introducing an infinite set of 'time' variables one can also treat the integrable equations as flows on infinite-dimensional Grassmannian manifolds [11]. Moreover, this theory reveals deep interrelations between the Hamiltonian structures of the KP hierarchy and two-dimensional conformal field theory as well as $W_{1+\infty}(\hat{W}_\infty)$ algebras [12, 13].

Further, the dispersionless versions of integrable hierarchies (KP, Toda, etc) have been considered [14–16]. These hierarchies can be understood as the quasi-classical limit $\hbar \to 0$ of the hierarchies described by the standard Lax equations. In this case, the pseudo-differential operators are replaced with a formal Laurent series in some variable λ and the commutators are replaced with the standard Poisson brackets.

Recently Strachan [17] has constructed the KP-type hierarchy arising from the Lax equation defined by the Moyal bracket. Deriving some first flows, he has suggested that the Moyal KP *hierarchy is closely related to the standard* KP *hierarchy considered in* the Sato approach. However, his paper does not contain the proof of the above conjecture.

In this paper we would like to show that for $k = 1/2$ these hierarchies are isomorphic. Comparing the appropriate flow equations we derive the general formula describing this isomorphism in the explicit form. As a simple application of these results, we obtain the conserved densities of the dispersionless KP equation.

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‡ Supported by KBN.

0305-4470/95/154381+08$19.50

2. The Sato KP and the Moyal KP hierarchies

Let us recall that the standard KP hierarchy (referred further as the Sato KP or $(\mathrm{KP})_s$) can be described as Lax equations [1]:

$$\frac{\partial \mathrm{L}}{\partial t_m} = [(\mathrm{L}^m)_+, \mathrm{L}] \qquad m = 1, 2, 3, \ldots \tag{2.1}$$

where L is the pseudo-differential operator

$$\mathrm{L} = \sum_{n=-2}^{\infty} v_n(\tilde{x})\partial_x^{-n-1} \tag{2.2}$$

$v_{-2}(\tilde{x}) \equiv 1$, $v_{-1}(\tilde{x}) \equiv 0$ and $\tilde{x} \equiv (t_1, t_2, t_3, \ldots.)$ is an infinite system of time variables (conventionally one can note that $t_1 \equiv x$, $t_2 \equiv y$ and $t_3 \equiv t$) and $(\mathrm{L}^m)_+$ denotes the differential part of the mth power of the operator (2.2).

The $(\mathrm{KP})_\mathrm{S}$ hierarchy (2.1) is equivalent to a system of evolution equations

$$\frac{\partial v_n}{\partial t_m} = f_n \tag{2.3}$$

for the infinitely many functions $v_n(\tilde{x})$ (coefficients of the operator (2.2)). The f_n are universal differential polynomials in the v_n and homogeneous of weight $m + n + 1$ if we give $\partial_x^j v_n(\tilde{x})$ weight $n + j + 1$.

The Moyal KP hierarchy [17] ($(\mathrm{KP})_\mathrm{M}$) can be introduced by replacing the operator L (2.2) with the operator of the form

$$\Lambda = \sum_{n=-2}^{\infty} u_n(\tilde{x})\lambda^{-n-1} \qquad (u_2(\tilde{x}) \equiv 1, u_{-1}(\tilde{x}) \equiv 0) \tag{2.4}$$

and by replacing the commutator [,] (in the Lax equations (2.1)) with the Moyal bracket

$$\{f, g\}_k = \sum_{s=0}^{\infty} \frac{k^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_\lambda^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_\lambda^j g). \tag{2.5}$$

As a result we obtain the system of nonlinear equations

$$\frac{\partial \Lambda}{\partial t_m} = \{(\Lambda^m)_+, \Lambda\}_k \qquad m = 1, 2, 3, \ldots. \tag{2.6}$$

The mth power of Λ is defined in terms of an associative $*$ product as

$$f * g = \sum_{s=0}^{\infty} \frac{k^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^j \partial_\lambda^{s-j} f)(\partial_x^{s-j} \partial_\lambda^j g) \tag{2.7}$$

such that

$$(\Lambda^m)_+ \equiv (\Lambda * \Lambda * \ldots * \Lambda)_+. \tag{2.8}$$

Considering the limit $k \to 0$, one can see that the Moyal bracket (2.5) collapses to the standard Poisson bracket

$$\lim_{k\to 0}\{f,g\}_k = \{f,g\} \tag{2.9}$$

where

$$\{f,g\} = \frac{\partial f}{\partial\lambda}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial\lambda}\frac{\partial f}{\partial x}. \tag{2.10}$$

In this case, from the $(\mathrm{KP})_{\mathrm{M}}$ hierarchy (2.6) we obtain the so-called dispersionless KP hierarchy [16]

$$\frac{\partial\Lambda}{\partial t_m} = \{(\Lambda^m)_+, \Lambda\}. \tag{2.11}$$

3. The relation between the Sato and the Moyal KP hierarchies

In this section we will prove the assumption (suggested first in [17]) that the Moyal KP hierarchy (2.6) is isomorphic with the hierarchy described by the formula (2.1).

First, let us consider the case $m = 2$. It is easy to see that $\mathrm{L}_+^2 = \partial_x^2 + 2v_0$ and that the KP flows (2.1) can be written as

$$\sum_{n=-2}^{\infty}(\partial_{t_2}v_n)\partial^{-n-1} = \left[\partial_x^2 + 2v_0, \sum_{n=-2}^{\infty} v_n\partial^{-n-1}\right]. \tag{3.1}$$

Computing the commutator (3.1), we obtain the formula

$$\partial_{t_2}v_n = v_n'' + v_{n+1}' - 2\sum_{i=1}^{n}(-1)^i\binom{n}{i}v_{n-i}v_0^{(i)} \qquad (n = 0, 1, 2, \ldots)$$
$$(v' \equiv \partial_x v(x) \text{ and } v^{(i)} \equiv \partial_x^i v(x)). \tag{3.2}$$

Alternatively, it follows from (2.6) that the appropriate $(\mathrm{KP})_{\mathrm{M}}$ flows have the form

$$\sum_{n=-2}^{\infty}(\partial_{t_2}u_n)\lambda^{-n-1} = \left\{\lambda^2 + 2u_0, \sum_{n=-2}^{\infty} u_n\lambda^{-n-1}\right\}_k. \tag{3.3}$$

Using definition (2.5), one can check that expression (3.3) is equivalent to the equation

$$\partial_{t_2}u_n = 2u_{n+1}' - 2\sum_{s=0}k^{2s}\binom{2s-n}{2s+n}u_0^{(2s+1)}u_{n-2s-1} \tag{3.4}$$

where $n = 0, 1, 2, \ldots$ and $n - 2s - 1 \geqslant 0$. Let us assume that the relation between the coefficients $u(x)$ and $v(x)$ is given by the general formula

$$u_n = \sum_{j=0}^{n}\beta(n,j)v_{n-j}^{(j)} \qquad n = 0, 1, 2, \ldots, m \tag{3.5}$$

where $\beta(n, j)$ denotes some constants. Substituting (3.5) into (3.4) we obtain

$$\sum_{j=0}^{n}\beta(n,j)(\partial_{t_2}v_{n-j}^{(j)}) = 2\sum_{j=0}^{n+1}\beta(n+1,j)v_{n+1-j}^{(j+1)}$$
$$-2\sum_{s=0}\sum_{j=0}^{n-2s-1}k^{2s}\binom{2s-n}{2s+1}\beta(n-2s-1,j)v_0^{(2s+1)}v_{n-2s-j-1}. \tag{3.6}$$

Next, using equations (3.6) and (3.2) we obtain the expression

$$\sum_{j=0}^{n}\beta(n,j)\left[v_{n-j}^{(j+2)} + 2v_{n-j+1}^{(j+1)} - 2\sum_{i=1}^{n-j}(-1)^i\binom{n-j}{i}(v_{n-j-i}v_0^{(i)})^{(j)}\right]$$
$$= 2\sum_{j=0}^{n+1}\beta(n+1,j)v_{(n+1-j)}^{(j+1)}$$
$$-2\sum_{s=0}\sum_{j=0}^{n-2s-1}k^{2s}\binom{2s-n}{2s+1}\beta(n-2s-1,j)v_0^{(2s+1)}v_{n-2s-j-1}^{(j)}. \tag{3.7}$$

A special situation arises when $k = 1/2$, in which case all the nonlinear terms (in (3.7)) cancel. Thus, assuming that $k = 1/2$, we obtain the following recurrence relation describing the numbers $\beta(n, j)$:

$$\sum_{j=0}^{n}\beta(n,j)[v_{n-j}^{(j+2)} + 2v_{n-j+1}^{(j+1)}] = 2\sum_{j=0}^{n+1}\beta(n+1,j)v_{n+1-j}^{(j+1)} \tag{3.8a}$$

or

$$\beta(n+1,j) = \beta(n,j) + \tfrac{1}{2}\beta(n,j-1) \tag{3.8b}$$

where $n, j = 1, 2, \dots$ and $\beta(0,0) = 1$.

Taking into account formulae (3.5) and (3.8), one can show that

$$\begin{array}{lllll}
\beta(1,0)=1 & \beta(1,1)=\frac{1}{2} & & & \\
\beta(2,0)=1 & \beta(2,1)=1 & \beta(2,2)=\frac{1}{4} & & \\
\beta(3,0)=1 & \beta(3,1)=\frac{3}{2} & \beta(3,2)=\frac{3}{4} & \beta(3,3)=\frac{1}{8} & \\
\beta(4,0)=1 & \beta(4,1)=2 & \beta(4,2)=\frac{3}{2} & \beta(4,3)=\frac{1}{2} & \beta(4,4)=\frac{1}{16}
\end{array} \tag{3.9}$$

etc and

$$\begin{aligned}
u_0 &= v_0 \\
u_1 &= v_1 + \tfrac{1}{2}\partial_x v_0 \\
u_2 &= v_2 + \partial_x v_1 + \tfrac{1}{4}\partial_x^2 v_0 \\
u_3 &= v_3 + \tfrac{3}{2}\partial_x v_2 + \tfrac{3}{4}\partial_x^2 v_1 + \tfrac{1}{8}\partial_x^3 v_0 \\
u_4 &= v_4 + 2\partial_x v_3 + \tfrac{3}{2}\partial_x^2 v_2 + \tfrac{1}{2}\partial_x^3 v_1 + \tfrac{1}{16}\partial_x^4 v_0
\end{aligned} \tag{3.10}$$

etc.

Generalizing this observation, we have the following proposition.

Proposition. The $(\mathrm{KP})_\mathrm{M}$ hierarchy (2.6) is isomorphic with the $(\mathrm{KP})_\mathrm{S}$ hierarchy (2.1) if and only if $k = 1/2$. The appropriate change of variables that establishes this isomorphism is given by the formula

$$u_n = \sum_{j=0}^{n} 2^{-j} \binom{n}{j} v_{n-j}^{(j)} \tag{3.11}$$

where $n = 0, 1, 2, \dots$ and $v^{(j)} \equiv \partial_x^j v$.

Remark. A calculation shows that for the arbitrary orders ($m \in N$) of the flows of the hierarchies (2.1) and (2.6), the relation (3.11) is the same. (In this paper we do not consider the problem of the rescaling of the variables x (generally a change of the basis) because it leads to new information about the algebraic properties of the Moyal KP hierarchy which we will consider in another paper. The author wishes to thank the referees for their comments about this point.)

4. Conserved densities of the dispersionless KP hierarchy

Let us consider an example of the application of the proposition presented in the previous section.

It is well known [2] that the $(\mathrm{KP})_\mathrm{s}$ hierarchy (2.1) can be derived from the compatibility condition of the linear system

$$\mathrm{L}\psi = \mu\psi \tag{4.1}$$

$$\frac{\partial}{\partial t_m}\psi = (\mathrm{L}^m)_+\psi \tag{4.2}$$

where the operators L and $(\mathrm{L}^m)_+$ are defined by (2.2) and (2.1) respectively; μ denotes the spectral parameter and ψ some eigenfunction.

Following the method described in [18], let us expand the quantity $\partial/\partial x$ in powers of the operator L:

$$\partial/\partial x = \mathrm{L} + \sigma_1(\tilde{x})\mathrm{L}^{-1} + \sigma_2(\tilde{x})\mathrm{L}^{-2} + \sigma_3(\tilde{x})\mathrm{L}^{-3} + \cdots. \tag{4.3}$$

Then, the coefficients $\sigma_j(\tilde{x})$ are determined by comparing equation (4.3) with (2.2). Applying equation (4.3) on the eigenfunction ψ and using equation (4.1) we obtain

$$\frac{\partial}{\partial x}\psi = \mu\psi + \sum_{j=1}^{\infty} \frac{\sigma_j(\tilde{x})}{\mu^j}\psi \tag{4.4}$$

which gives

$$\sum_{j=1}^{\infty} \frac{\sigma_j(\tilde{x})}{\mu^j} = \frac{\partial}{\partial x}\log\psi - \mu. \tag{4.5}$$

Differentiating equation (4.5) with respect to the parameters $\tilde{x} \equiv \{t_m, m = 1, 2, \ldots\}$ we obtain

$$\frac{\partial}{\partial t_m}\left(\sum_{j=1}^{\infty} \frac{\sigma_j(\tilde{x})}{\mu^j}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t_m} \log \psi\right). \tag{4.6}$$

Thus, we can see that each quantity $\sigma_j(\tilde{x})$ gives a conserved density of the $(\mathrm{KP})_{\mathrm{S}}$ hierarchy (2.1).

In order to find the conserved quantities of the dispersionless KP, first we have to compute the $\sigma_j[v]$'s and then (using the formula (3.11)) we can rewrite it in the terms of the u coefficients:

$$\begin{aligned}
\sigma_1[u] &= -u_0 \\
\sigma_2[u] &= u_1 - \tfrac{1}{2}\partial_x u_0 \\
\sigma_3[u] &= -u_2 - u_0^2 + \partial_x u_1 - \tfrac{1}{4}\partial_x^2 u_0 \\
\sigma_4[u] &= -u_3 - 3u_2 u_1 + \tfrac{3}{2}\partial_x u_2 + \tfrac{3}{2}u_2(\partial_x u_0) + 3u_1(\partial_x u_1) + \tfrac{1}{2}\partial_x(u_0^2) - \tfrac{3}{2}(\partial_x u_0)(\partial_x u_1) \\
&\quad - \tfrac{3}{4}\partial_x^2 u_1 - \tfrac{3}{4}u_1(\partial_x^2 u_0) + \tfrac{3}{8}(\partial_x u_0)(\partial_x^2 u_0) + \tfrac{1}{8}\partial_x^3 u_0 \\
\sigma_5[u] &= -u_4 - 4u_3 u_0 - 2u_1^2 - 2u_0^3 + 2\partial_x u_3 + 5u_1(\partial_x u_0) + u_0(\partial_x u_1) + 6u_0(\partial_x u_2) \\
&\quad - \tfrac{3}{2}\partial_x^2 u_2 - 3u_0(\partial_x^2 u_1) - 2(\partial_x u_0)^2 - \tfrac{1}{2}u_0(\partial_x^2 u_0) - (\partial_x u_0)(\partial_x^2 u_0) + \tfrac{1}{2}\partial_x^3 u_1 \\
&\quad + \tfrac{1}{2}u_0(\partial_x^3 u_0) - \tfrac{1}{16}(\partial_x^4 u_0)
\end{aligned} \tag{4.7}$$

etc.

Next, we should calculate the $(\mathrm{KP})_{\mathrm{M}}$ flows $(\partial_{t_2} u_n, \partial_{t_3} u_n, \ldots)$ (assuming the arbitrary value of k) and then inserting it into the (4.7) we obtain the relations which depend on the parameter k. Finally, taking the limit $k \to 0$, we obtain the conserved densities corresponding to the dynamical equations which are included in the dispersionless KP hierarchy.

This may be summarized by the following diagram.

$$\begin{array}{ccccc}
(\mathrm{KP})_{\mathrm{S}} & \overset{\text{isomorph.}}{\underset{}{\overset{(3.11)}{\longleftrightarrow}}} & (\mathrm{KP})_{\mathrm{M}} & \overset{\text{limit}}{\overset{k\to 0}{\longrightarrow}} & (\mathrm{KP})_{\mathrm{D}} \\
\Big| & & \Big| & & \\
\overset{\text{cd}}{(\mathrm{KP})}_{\mathrm{S}} & \overset{\text{isomorph.}}{\overset{(3.11)}{\longleftrightarrow}} & \overset{\text{cd}}{(\mathrm{KP})}_{\mathrm{M}} & \overset{\text{limit}}{\overset{k\to 0}{\longrightarrow}} & \overset{\text{cd}}{(\mathrm{KP})}_{\mathrm{D}}
\end{array}$$

where the symbol cd denotes the conserved densities and $(\mathrm{KP})_{\mathrm{D}}$ denotes the dispersionless KP hierarchy.

As an example, let us consider the $(\mathrm{KP})_{\mathrm{M}}$ equation

$$\tfrac{3}{4}\partial_y^2 u_0 = \partial_x(\partial_t u_0 - 3u_0 \partial_x u_0 - k^2 \partial_x^3 u_0) \tag{4.8}$$

such that for $k = 1/2$ it is the standard KP [2], and the dispersionless limit corresponds to $k \to 0$. Because equation (4.8) depends only on the one variable u_0, we also have to eliminate $u_1, u_2, \ldots$ from relations (4.7). To achieve this we can use the constraints arising from the $m = 2$ $(\mathrm{KP})_{\mathrm{M}}$ flows. Indeed, it follows from (3.4) that

$$\begin{aligned}
\partial_y u_0 &= 2\partial_x u_1 \\
\partial_y u_1 &= 2\partial_x u_2 + \partial_x(u_0^2) \\
\partial_y u_2 &= 2\partial_x u_3 + 4u_1(\partial_x u_0) \\
\partial_y u_3 &= 2\partial_x u_4 + 6u_2(\partial_x u_0) + 2k^2 u_0(\partial_x^3 u_0) \\
\partial_y u_4 &= 2\partial_x u_5 + 8u_3(\partial_x u_0) + 8k^2 u_1(\partial_x^3 u_0)
\end{aligned} \tag{4.9}$$

etc thus, we have

$$\begin{aligned}
u_1 &= \tfrac{1}{2}\partial_x^{-1}(\partial_y u_0) \\
u_2 &= -\tfrac{1}{2}u_0^2 + \tfrac{1}{4}\partial_x^{-2}(\partial_y u_0) \\
u_3 &= -\tfrac{1}{4}\partial_x^{-1}(\partial_y u_0^2) - \partial_x^{-1}\{[\partial_x^{-1}(\partial_y u_0)](\partial_x u_0)\} \\
u_4 &= \tfrac{3}{2}\partial_x^{-1}[u_0^2(\partial_x u_0)] - \tfrac{3}{4}\partial_x^{-1}\{(\partial_x u_0)[\partial_x^{-2}(\partial_y u_0)]\} - \tfrac{1}{8}\partial_x^{-2}(\partial_y^2 u_0^2) - \tfrac{1}{2}\partial_x^{-2}\partial_y\{[\partial_x^{-1}(\partial_y u_0)]\} \\
&\quad + \tfrac{1}{16}\partial_x^{-4}(\partial_y^3 u_0) - k^2\partial_x^{-1}[(\partial_x^3 u_0)u_0]
\end{aligned} \tag{4.10}$$

etc where ∂_x^{-1} denotes the integration with respect to x. Substitution of equations (4.10) into (4.7) gives

$$\begin{aligned}
\sigma_1 &= -u_0 \\
\sigma_2 &= -\tfrac{1}{2}(\partial_x u_0) + \tfrac{1}{2}\partial_x^{-1}(\partial_y u_0) \\
\sigma_3 &= -\tfrac{1}{2}u_0^2 + \tfrac{1}{2}(\partial_y u_0) - \tfrac{1}{4}(\partial_x^2 u_0) - \tfrac{1}{4}\partial_0^{-2}(\partial_y u_0) \\
\sigma_4 &= \tfrac{1}{8}(\partial_x^3 u_0) - \tfrac{1}{4}(\partial_x u_0^3) - \tfrac{1}{4}(\partial_x u_0^2) - \tfrac{3}{8}(\partial_x\partial_y u_0) + \tfrac{3}{8}(\partial_x u_0)(\partial_x^2 u_0) - \tfrac{3}{4}(\partial_y u_0)(\partial_x u_0) \\
&\quad - \tfrac{1}{4}\partial_x^{-1}(\partial_y u_0^2) + \tfrac{3}{4}[\partial_x^{-1}(\partial_y u_0)][\tfrac{1}{2} + u_0^2 + \partial_y u_0 - \tfrac{1}{2}\partial_x^2 u_0] \\
&\quad - \partial_x^{-1}\{[\partial_x^{-1}(\partial_y u_0)](\partial_x u_0)\} + \tfrac{3}{8}[\partial_x^{-2}(\partial_y u_0)][\partial_x u_0 - \partial_x^{-1}(\partial_y u_0)] \\
&\quad + \tfrac{1}{8}\partial_x^{-3}(\partial_y^2 u_0) \\
\sigma_5 &= -2u_0^3 - \tfrac{1}{4}\partial_y(u_0^2) - \tfrac{3}{8}\partial_y u_0 - \tfrac{1}{2}(\partial_x u_0)^2 + u_0(\partial_x^2 u_0) - (\partial_x u_0)(\partial_x^2 u_0) + \tfrac{1}{2}u_0(\partial_3^3 u_0) \\
&\quad - \tfrac{1}{16}(\partial_x^4 u_0) + \tfrac{1}{4}\partial_y\partial_x^2 u_0 - 3u_0[\partial_x(u_0^2)] - \tfrac{3}{2}u_0(\partial_x\partial_y u_0) + \tfrac{3}{2}u_0[\partial_x^{-1}(\partial_y u_0)] \\
&\quad - \tfrac{1}{2}[\partial_x^{-1}(\partial_y u_0)]^2 + [\partial_x^{-1}(\partial_y u_0^2)]u_0 + \tfrac{3}{2}\partial_x^{-1}[u_0^2(\partial_x u_0)] \\
&\quad - \tfrac{3}{4}\partial_x^{-1}\{(\partial_x u_0)[\partial_x^{-2}(\partial_y u_0)]\} + (4\partial_x^{-1} - \tfrac{1}{2}\partial_x^{-2}\partial_y + \tfrac{1}{2})[\partial_x^{-1}(\partial_y u_0)](\partial_x u_0) \\
&\quad - \tfrac{1}{8}\partial_x^{-2}(\partial_y^2 u_0^2) + \tfrac{1}{16}\partial_x^{-4}(\partial_y^3 u_0) - k^2\partial_x^{-1}[(\partial_x^3 u_0)u_0]
\end{aligned} \tag{4.11}$$

etc.

Using (3.11) one can check that for $k = 1/2$ formulae (4.11) describe the example of the conserved densities of the KP equation (4.8). The dispersionless version of these densities can be obtained by considering the limit $k \to 0$.

5. Final remarks

In this paper we have proved that the Moyal KP and the Sato KP hierarchies are isomorphic. It would be interesting to use this fact to construct the Moyal version of other integrable hierarchies, e.g. the fractional KP–KDV [5], the continuous KP [19], the generalized Drinfeld–Sokolov [6] etc. In this way one can try to define the dispersionless limit of the above systems. Another question that arises naturally in this context is to find whether the Gelfand–Dickey Hamiltonian structures [1] can be described by the Moyal bracket approach.

Acknowledgments

The author wishes to thank Z Popowicz and M Widawska for fruitful discussions and for their valuable comments.

References

[1] Dickey L A 1991 *Soliton Equations and Hamiltonian Systems (Advanced Series in Math. Phys. 12)* (Singapore: World Scientific)
[2] Date E, Jimbo M, Kashiwara M and Miwa T 1983 *Proc. of RIMS Symp. on Nonlinear Integrable Systems* ed M Jimbo and T Miwa (Singapore: World Scientific)
[3] Manin Yu and Radul A 1985 *Comm. Math. Phys.* **98** 65
[4] Rabin J M 1991 *Comm. Math. Phys.* **137** 533
[5] Khesin B and Zakharevich I 1993 *C. R. Acad. Sci. I* **316** 621
[6] de Groot M F, Hollowood T J and Luis Miramontes J 1992 *Comm. Math. Phys.* **145** 57
[7] Sato M and Sato Y 1983 *Nonlinear Partial Differential Equations in Applied Science* ed H Fujita, P D Lax and G Strang (Tokyo: Kinokuniya/North-Holland) p 259
[8] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[9] Konopelchenko B and Oevel W 1993 *Publ. RIMS. Kyoto Univ.* **29** 581
[10] Adler M 1979 *Invent. Math.* **50** 219
[11] Jimbo M and Miwa T 1983 *Publ. RIMS. Kyoto Univ.* **19** 943
[12] Yamagishi K 1991 *Phys. Lett.* **259B** 436
[13] Yu F and Wu Y-S 1992 *Nucl. Phys.* B **373** 713
[14] Kodama Y 1988 *Phys. Lett.* **129A** 223; 1990 *Phys. Lett.* **147A** 477
[15] Kodama Y and Gibbons J 1989 *J. Phys. Lett.* A **135** 167
[16] Takasaki K and Takebe T 1991 *Lett. Math. Phys.* **23** 205; 1993 *Lett. Math. Phys.* **28** 165
[17] Strachan I A B 1994 The Moyal bracket and the dispersionless limit of the KP hierarchy *Preprint* Newcastle University
[18] Drinfeld V and Sokolov V 1985 *J. Sov. Math.* **30** 1975
Matsukidaira J, Satsuma J and Strampp W 1990 *J. Math. Phys.* **31** 1426
[19] Gawrylczyk J 1995 *J. Math. Phys.* **36** to appear